

# DEFORMATIONS OF THE PICARD BUNDLE

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**ABSTRACT.** Let  $X$  be a nonsingular algebraic curve of genus  $g \geq 3$ , and let  $\mathcal{M}_\xi$  denote the moduli space of stable vector bundles of rank  $n \geq 2$  and degree  $d$  with fixed determinant  $\xi$  over  $X$  such that  $n$  and  $d$  are coprime and  $d > n(2g - 2)$ . We assume that if  $g = 3$  then  $n \geq 4$  and if  $g = 4$  then  $n \geq 3$ . Let  $\mathcal{W}_\xi(L)$  denote the vector bundle over  $\mathcal{M}_\xi$  defined by the direct image  $p_{\mathcal{M}_\xi*}(\mathcal{U}_\xi \otimes p_X^* L)$  where  $\mathcal{U}_\xi$  is a universal vector bundle over  $X \times \mathcal{M}_\xi$  and  $L$  is a line bundle over  $X$  of degree zero. The space of infinitesimal deformations of  $\mathcal{W}_\xi(L)$  is proved to be isomorphic to  $H^1(X, \mathcal{O}_X)$ . This construction gives a complete family of vector bundles over  $\mathcal{M}_\xi$  parametrized by the Jacobian  $J$  of  $X$  such that  $\mathcal{W}_\xi(L)$  is the vector bundle corresponding to  $L \in J$ . The connected component of the moduli space of stable sheaves with the same Hilbert polynomial as  $\mathcal{W}_\xi(\mathcal{O})$  over  $\mathcal{M}_\xi$  containing  $\mathcal{W}_\xi(\mathcal{O})$  is in fact isomorphic to  $J$  as a polarised variety.

## 1. INTRODUCTION

Let  $X$  be a connected nonsingular projective algebraic curve of genus  $g \geq 2$  defined over the complex numbers. Let  $J$  denote the Jacobian (Picard variety) of  $X$  and  $J^d$  the variety of line bundles of degree  $d$  over  $X$ ; thus in particular  $J^0 = J$ . Suppose  $d \geq 2g - 1$  and let  $\mathcal{L}$  be a Poincaré (universal) bundle over  $X \times J^d$ . If we denote by  $p_J$  the natural projection from  $X \times J^d$  to  $J^d$ , the direct image  $p_{J*}\mathcal{L}$  is then locally free and is called the Picard bundle of degree  $d$ .

These bundles have been investigated by a number of authors over at least the last 40 years. It may be noted that the projective bundle corresponding to  $p_{J*}\mathcal{L}$  can be identified with the  $d$ -fold symmetric product  $S^d(X)$ . Picard bundles were studied in this light by A. Mattuck [12, 13] and I. G. Macdonald [14] among others; both Mattuck and Macdonald gave formulae for their Chern classes. Somewhat later R. C. Gunning [7, 8] gave a more analytic treatment involving theta-functions. Later still, and of especial relevance to us, G. Kempf [10] and S. Mukai [15] independently studied the deformations of the Picard bundle; the problem then is to obtain an inversion formula showing that all deformations of  $p_{J*}\mathcal{L}$  arise in a natural way. Kempf and Mukai proved that  $p_{J*}\mathcal{L}$  is simple and that, if  $X$  is not hyperelliptic, the space of

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1991 *Mathematics Subject Classification.* 14H60, 14J60.

All authors are members of the research group VBAC (Vector Bundles on Algebraic Curves), which is partially supported by EAGER (EC FP5 Contract no. HPRN-CT-2000-00099) and by EDGE (EC FP5 Contract no. HPRN-CT-2000-00101). The second author acknowledges the support of CONACYT grant 28492-E.

infinitesimal deformations of  $p_{J*}\mathcal{L}$  has dimension given by

$$\dim H^1(J^d, \text{End}(p_{J*}\mathcal{L})) = 2g.$$

Moreover, all the infinitesimal deformations arise from genuine deformations. In fact there is a complete family of deformations of  $p_{J*}\mathcal{L}$  parametrised by  $J \times \text{Pic}^0(J^d)$ , the two factors corresponding respectively to translations in  $J^d$  and deformations of  $\mathcal{L}$  ([10, §9], [15, Theorem 4.8]). (The deformations of  $\mathcal{L}$  are given by  $\mathcal{L} \mapsto \mathcal{L} \otimes p_J^*L$  for  $L \in \text{Pic}^0(J^d)$ .) Since  $J$  is a principally polarised abelian variety and  $J^d \cong J$ ,  $\text{Pic}^0(J^d)$  can be identified with  $J$  (strictly speaking  $\text{Pic}^0(J^d)$  is the dual abelian variety, but the principal polarisation allows the identification).

Mukai's paper [15] was set in a more general context involving a transform which provides an equivalence between the derived category of the category of  $\mathcal{O}_A$ -modules over an abelian variety  $A$  and the corresponding derived category on the dual abelian variety  $\hat{A}$ . This technique has come to be known as the Fourier–Mukai transform and has proved very useful in studying moduli spaces of sheaves on abelian varieties and on some other varieties.

Our object in this paper is to generalise the results of Kempf and Mukai on deformations of Picard bundles to the moduli spaces of higher rank vector bundles over  $X$  with fixed determinant. In particular we compute the space of infinitesimal deformations of the Picard bundle in this context and also identify a complete family of deformations. While the analogy is not precise, this identification can be seen as a type of Fourier–Mukai transform.

We fix a holomorphic line bundle  $\xi$  over  $X$  of degree  $d$ . Let  $\mathcal{M}_\xi := \mathcal{M}_\xi(n, d)$  be the moduli space of stable vector bundles  $E$  over  $X$  with  $\text{rank}(E) = n \geq 2$ ,  $\deg(E) = d$  and  $\bigwedge^n E = \xi$ . We assume that  $n$  and  $d$  are coprime, ensuring the smoothness and completeness of  $\mathcal{M}_\xi$ , and that  $g \geq 3$ . We assume also that if  $g = 3$  then  $n \geq 4$  and if  $g = 4$  then  $n \geq 3$ . The case  $g = 2$  together with the three special cases  $g = 3$  with  $n = 2, 3$  and  $g = 4$  with  $n = 2$  are omitted in our main result since the method of proof does not cover these cases.

It is known that there is a universal vector bundle over  $X \times \mathcal{M}_\xi$ . Two such universal bundles differ by tensoring with the pullback of a line bundle on  $\mathcal{M}_\xi$ . However, since  $\text{Pic}(\mathcal{M}_\xi) = \mathbb{Z}$ , it is possible to choose canonically a universal bundle. Let  $l$  be the smallest positive number such that  $ld \equiv 1 \pmod{n}$ . There is a unique universal vector bundle  $\mathcal{U}_\xi$  over  $X \times \mathcal{M}_\xi$  such that  $\bigwedge^n \mathcal{U}_\xi|_{\{x\} \times \mathcal{M}_\xi} = \Theta^{\otimes l}$  [18], where  $x \in X$  and  $\Theta$  is the ample generator of  $\text{Pic}(\mathcal{M}_\xi)$ . Henceforth, by a universal bundle we will always mean this canonical one. We denote by  $p_X$  and  $p_{\mathcal{M}}$  the natural projections of  $X \times \mathcal{M}_\xi$  onto the two factors.

Now suppose that  $d > n(2g - 2)$ . For any  $L \in J$ , let

$$\mathcal{W}_\xi(L) := p_{\mathcal{M}*}(\mathcal{U}_\xi \otimes p_X^*L)$$

be the direct image. The assumption on  $d$  ensures that  $\mathcal{W}_\xi(L)$  is a locally free sheaf on  $\mathcal{M}_\xi$  and all the higher direct images of  $\mathcal{U}_\xi \otimes p_X^* L$  vanish. The rank of  $\mathcal{W}_\xi(L)$  is  $d + n(1 - g)$  and  $H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(L)) \cong H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L)$ . By analogy with the case  $n = 1$ , we shall refer to the bundles  $\mathcal{W}_\xi(L)$  as *Picard bundles*.

Our first main result concerns the infinitesimal deformations of  $\mathcal{W}_\xi(L)$ . We prove

**Theorem 2.9.** *For any line bundle  $L \in J$ , the space of infinitesimal deformations of the vector bundle  $\mathcal{W}_\xi(L)$ , namely  $H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(L)))$ , is canonically isomorphic to  $H^1(X, \mathcal{O}_X)$ . In particular,*

$$\dim H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(L))) = g.$$

In the special case where  $n = 2$  and  $L = \mathcal{O}_X$ , this was proved by V. Balaji and P. A. Vishwanath in [1] using a construction of M. Thaddeus [19]. For all  $n$ , it is known that  $\mathcal{W}_\xi(\mathcal{O}_X)$  is simple [5] and indeed that it is stable (with respect to the unique polarisation of  $\mathcal{M}_\xi$ ) [4]; in fact the proof of stability generalises easily to show that  $\mathcal{W}_\xi(L)$  is stable. In this context, note that Y. Li [11] has proved a stability result for Picard bundles over the non-fixed determinant moduli space  $\mathcal{M}(n, d)$ , but this does not imply the result for  $\mathcal{M}_\xi$ . As a byproduct of our proof of Theorem 2.9, we obtain a new proof that  $\mathcal{W}_\xi(L)$  is simple (Corollary 2.8).

We can consider the bundles  $\{\mathcal{W}_\xi(L)\}$  as a family of bundles over  $\mathcal{M}_\xi$ , parametrized by  $J$ . We prove that this is a complete family of deformations, both globally and in the local sense that the infinitesimal deformation map

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(L)))$$

is an isomorphism for all  $L$  (Corollary 5.2). As we noted, the bundles  $\mathcal{W}_\xi(L)$  are stable. We denote by  $\mathcal{M}_X^0(\mathcal{W}_\xi)$  the connected component of the moduli space of stable sheaves with the same Hilbert polynomial as  $\mathcal{W}_\xi(\mathcal{O})$  on  $\mathcal{M}_\xi$  containing  $\mathcal{W}_\xi(\mathcal{O})$ . We prove

**Theorem 5.1.** *The morphism*

$$\phi : J \longrightarrow \mathcal{M}_X^0(\mathcal{W}_\xi)$$

*given by  $\phi(L) = \mathcal{W}_\xi(L)$  is an isomorphism of polarised varieties.*

A further consequence is the following Torelli theorem: if  $X$  and  $X'$  are smooth projective curves,  $\xi, \xi'$  are line bundles of degree  $d$  over  $X, X'$  respectively and  $\mathcal{M}_X^0(\mathcal{W}_\xi) \cong \mathcal{M}_{X'}^0(\mathcal{W}_{\xi'})$  as polarised varieties, then  $X \cong X'$  (Corollary 5.4).

*Notation and assumptions.* We work throughout over the complex numbers and suppose that  $X$  is a connected nonsingular projective algebraic curve of genus  $g \geq 3$ . We suppose that  $n \geq 2$  and that if  $g = 3$  then  $n \geq 4$  and if  $g = 4$  then  $n \geq 3$ . We assume moreover that  $(n, d) = 1$  and  $d > n(2g - 2)$ . In general, we denote the natural

projections of  $X \times Y$  onto its factors by  $p_X, p_Y$ , writing  $p_{\mathcal{M}}$  for  $p_{\mathcal{M}_\xi}$  for simplicity. For a variety  $X \times Y \times Z$ , we denote by  $p_i$  ( $i = 1, 2, 3$ ) the projection onto the  $i$ -th factor and by  $p_{ij}$  the projection onto the Cartesian product of the  $i$ -th and the  $j$ -th factors. Finally, for any  $x \in X$ , we denote by  $\mathcal{U}_x$  the bundle over  $\mathcal{M}_\xi$  obtained by restricting  $\mathcal{U}_\xi$  to  $\{x\} \times \mathcal{M}_\xi$ .

*Acknowledgements.* Part of this work was done during a visit of the authors to ICTP, Italy. The authors thank ICTP for its hospitality. The third author would like also to thank the Isaac Newton Institute, Cambridge and the organisers of the HDG programme for their hospitality during the completion of work on this paper.

## 2. COHOMOLOGY OF $\mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*$

To compute the cohomology groups  $H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*)$  we need the following propositions.

**Proposition 2.1.** *If  $L_1$  and  $L_2$  are two line bundles in  $J$  then*

$$H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*) \cong H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{W}_\xi(L_2)^*)$$

for any  $i \geq 0$ . Moreover,

$$H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*) \cong H^{i+1}(X \times \mathcal{M}_\xi \times X, p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^* \otimes p_3^* L_2^* \otimes p_3^* K_X)$$

for  $i \geq 0$ , where  $K_X$  is the canonical line bundle over  $X$ .

*Proof.* By the assumption that  $d > n(2g - 2)$  we have that  $H^1(X, \mathcal{U}_\xi|_{X \times \{v\}} \otimes L_1) = 0$  for all  $v \in \mathcal{M}_\xi$ . Using the projection formula and the Leray spectral sequence we have

$$H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{W}_\xi(L_2)^*) \cong H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*).$$

This proves the first part.

For every stable vector bundle  $E$  of rank  $n$  and degree  $d$ ,

$$H^0(X, E^* \otimes L_2^* \otimes K_X) \cong H^1(X, E \otimes L_2)^* = 0.$$

Consequently, the projection formula gives

$$\mathcal{R}^i p_{12*}(p_{23}^*(\mathcal{U}_\xi^*) \otimes p_3^* L_2^* \otimes p_3^* K_X) = 0$$

for  $i \neq 1$ , and we have by relative Serre duality

$$\mathcal{R}^1 p_{12*}(p_{23}^*(\mathcal{U}_\xi^*) \otimes p_3^* L_2^* \otimes p_3^* K_X) \cong p_{\mathcal{M}}^* \mathcal{W}_\xi(L_2)^*,$$

Finally, using the projection formula and the Leray spectral sequence, it follows that

$$H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{W}_\xi(L_2)^*).$$

$$\cong H^{i+1}(X \times \mathcal{M}_\xi \times X, p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^* \otimes p_3^* L_2^* \otimes p_3^* K_X)$$

for  $i \geq 0$ . Thus,

$$H^{i+1}(X \times \mathcal{M}_\xi \times X, p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^* \otimes p_3^* L_2^* \otimes p_3^* K_X) \cong H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*)$$

as asserted in the proposition.  $\square$

**Remark 2.2.** Proposition 2.1 can be formulated in a more general context. Let  $V_1, V_2$  be flat families of vector bundles over  $X$  parametrised by a complete irreducible variety  $Y$  such that for each  $y \in Y$  we have  $H^1(X, V_i|_{X \times \{y\}}) = 0$  for  $i = 1, 2$ . Under this assumption

$$H^i(Y, p_{Y*}V_1 \otimes (p_{Y*}V_2)^*) \cong H^{i+1}(X \times Y \times X, p_{12}^*V_1 \otimes p_{23}^*V_2^* \otimes p_3^*K_X).$$

The proof is the same as for Proposition 2.1.

Denote by  $\mathcal{R}^i$  the  $i$ -th direct image of  $p_{12}^*\mathcal{U}_\xi \otimes p_1^*L_1 \otimes p_{23}^*\mathcal{U}_\xi^*$  for the projection  $p_3$ , that is

$$\mathcal{R}^i := \mathcal{R}^i p_{3*}(p_{12}^*\mathcal{U}_\xi \otimes p_1^*L_1 \otimes p_{23}^*\mathcal{U}_\xi^*).$$

From the Leray spectral sequence we obtain the exact sequences

$$\begin{aligned} 0 \longrightarrow H^1(X, \mathcal{R}^0 \otimes L_2^* \otimes K_X) &\longrightarrow H^1(X \times \mathcal{M}_\xi \times X, p_{12}^*\mathcal{U}_\xi \otimes p_1^*L_1 \otimes p_{23}^*\mathcal{U}_\xi^* \otimes p_3^*L_2^* \otimes p_3^*K_X) \\ (1) \qquad \qquad \qquad &\longrightarrow H^0(X, \mathcal{R}^1 \otimes L_2^* \otimes K_X) \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \longrightarrow H^1(X, \mathcal{R}^1 \otimes L_2^* \otimes K_X) &\longrightarrow H^2(X \times \mathcal{M}_\xi \times X, p_{12}^*\mathcal{U}_\xi \otimes p_1^*L_1 \otimes p_{23}^*\mathcal{U}_\xi^* \otimes p_3^*L_2^* \otimes p_3^*K_X) \\ (2) \qquad \qquad \qquad &\longrightarrow H^0(X, \mathcal{R}^2 \otimes L_2^* \otimes K_X) \longrightarrow 0. \end{aligned}$$

**Proposition 2.3.**  $\mathcal{R}^0 = 0$ .

*Proof.* Note that for any  $x \in X$

$$(3) \qquad H^0(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^*L_1 \otimes p_{\mathcal{M}}^*\mathcal{U}_x^*) \cong H^0(X, L_1 \otimes p_{X*}(\mathcal{U}_\xi \otimes p_{\mathcal{M}}^*\mathcal{U}_x^*)).$$

From [3] and [16] we know that for generic  $y \in X$ , the two vector bundles  $\mathcal{U}_y$  and  $\mathcal{U}_x$  are non-isomorphic and stable. Hence  $H^0(\mathcal{M}_\xi, \mathcal{U}_y \otimes \mathcal{U}_x^*) = 0$ . This implies that

$$(4) \qquad p_{X*}(\mathcal{U}_\xi \otimes p_{\mathcal{M}}^*\mathcal{U}_x^*) = 0.$$

So (3) gives

$$\mathcal{R}^0 = \mathcal{R}^0 p_{3*}(p_{12}^*\mathcal{U}_\xi \otimes p_1^*L_1 \otimes p_{23}^*\mathcal{U}_\xi^*) = 0$$

and the proof is complete.  $\square$

In the following sections we will use Hecke transformations to compute the fibre of  $\mathcal{R}^i$  for  $i = 1, 2$ , and will prove the following propositions.

**Proposition 2.4.**  $\mathcal{R}^2 = 0$ .

**Proposition 2.5.**  $\mathcal{R}^1$  is a line bundle. Moreover,  $\mathcal{R}^1 \cong L_1 \otimes TX$ .

Assume for the moment Propositions 2.4 and 2.5.

From the exact sequence (1) and the previous propositions we have the following theorem.

**Theorem 2.6.**  $H^0(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*) \cong H^0(X, L_1 \otimes L_2^*)$ .

*Proof.* Combining (1), Proposition 2.3 and the second part of Proposition 2.1 it follows that

$$H^0(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*) \cong H^0(X, \mathcal{R}^1 \otimes L_2^* \otimes K_X).$$

From Proposition 2.5 it follows immediately that

$$H^0(X, \mathcal{R}^1 \otimes L_2^* \otimes K_X) \cong H^0(X, L_1 \otimes L_2^*)$$

and hence the proof is complete.  $\square$

**Corollary 2.7.** *If  $L_1 \not\cong L_2$ , then  $\mathcal{W}_\xi(L_1) \not\cong \mathcal{W}_\xi(L_2)$ .*

*Proof.* Since  $L_1 \not\cong L_2$ , we have

$$H^0(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*) = 0.$$

Consequently,  $\mathcal{W}_\xi(L_1) \not\cong \mathcal{W}_\xi(L_2)$ .  $\square$

The following is also a corollary of Theorem 2.6.

**Corollary 2.8.** *The vector bundle  $\mathcal{W}_\xi(L)$  is simple for any  $L \in J$ . In other words,  $H^0(\mathcal{M}_\xi, \mathcal{W}_\xi(L) \otimes \mathcal{W}_\xi(L)^*) \cong \mathbb{C}$ .*

The following theorem gives the infinitesimal deformations of  $\mathcal{W}_\xi(L)$ .

**Theorem 2.9.** *For any line bundle  $L \in J$ , the space of infinitesimal deformations of the vector bundle  $\mathcal{W}_\xi(L)$ , namely  $H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(L)))$ , is canonically isomorphic to  $H^1(X, \mathcal{O}_X)$ . In particular,*

$$\dim H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(L))) = g.$$

*Proof.* Let  $L_1 = L_2 = L$ . From (2) and Proposition 2.4 we obtain an isomorphism

$$(5) \quad H^1(X, \mathcal{R}^1 \otimes L^* \otimes K_X) \cong H^2(X \times \mathcal{M}_\xi \times X, p_{12}^* \mathcal{U}_\xi \otimes p_1^* L \otimes p_{23}^* \mathcal{U}_\xi^* \otimes p_3^* L^* \otimes p_3^* K_X).$$

From Proposition 2.5 we have

$$\begin{aligned} H^1(X, \mathcal{R}^1 \otimes L^* \otimes K_X) &\cong H^1(X, L \otimes TX \otimes L^* \otimes K_X) \\ &\cong H^1(X, \mathcal{O}_X). \end{aligned}$$

Combining this observation with Proposition 2.1 and (5) we get

$$H^1(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(L))) \cong H^1(X, \mathcal{O}_X).$$

$\square$

**Remark 2.10.** From the proof of Theorem 2.9 and Proposition 2.1 we see that if  $L_1$  and  $L_2$  are not isomorphic then

$$H^1(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*) \cong H^1(X, L_1 \otimes L_2^*).$$

Hence

$$\dim H^1(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*) = g - 1.$$

### 3. THE HECKE TRANSFORMATION

In this section we will use Hecke transformations to compute the cohomology groups  $H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)$  for any  $x \in X$  and prove Proposition 2.4. The details of the Hecke transformation and its properties can be found in [16, 17]. We will briefly describe it and note those properties that will be needed here.

Fix a point  $x \in X$ . Let  $\mathbb{P}(\mathcal{U}_x)$  denote the projective bundle over  $\mathcal{M}_\xi$  consisting of lines in  $\mathcal{U}_x$ . If  $f$  denotes the natural projection of  $\mathbb{P}(\mathcal{U}_x)$  to  $\mathcal{M}_\xi$  and  $\mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(-1)$  the tautological line bundle then

$$f_* \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) \cong \mathcal{U}_x^*,$$

and  $\mathcal{R}^j f_* \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) = 0$  for all  $j > 0$ . From the commutative diagram

$$\begin{array}{ccc} X \times \mathbb{P}(\mathcal{U}_x) & \xrightarrow{\text{Id}_X \times f} & X \times \mathcal{M}_\xi \\ p_{\mathbb{P}(\mathcal{U}_x)} \downarrow & & \downarrow p_{\mathcal{M}} \\ \mathbb{P}(\mathcal{U}_x) & \xrightarrow{f} & \mathcal{M}_\xi \end{array}$$

and the base change theorem, we deduce that

$$\begin{aligned} & H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) \\ (6) \quad & \cong H^i(X \times \mathbb{P}(\mathcal{U}_x), (\text{Id}_X \times f)^*(\mathcal{U}_\xi \otimes p_X^* L_1) \otimes p_{\mathbb{P}(\mathcal{U}_x)}^* \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)) \end{aligned}$$

for all  $i$ .

Moreover, since  $p_{\mathbb{P}(\mathcal{U}_x)*}(\text{Id}_X \times f)^*(\mathcal{U}_\xi \otimes p_X^* L_1) \cong f^* \mathcal{W}_\xi(L_1)$ , there is a canonical isomorphism

$$\begin{aligned} & H^i(X \times \mathbb{P}(\mathcal{U}_x), (\text{Id}_X \times f)^*(\mathcal{U}_\xi \otimes p_X^* L_1) \otimes p_{\mathbb{P}(\mathcal{U}_x)}^* \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)) \\ (7) \quad & \cong H^i(\mathbb{P}(\mathcal{U}_x), f^* \mathcal{W}_\xi(L_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)) \end{aligned}$$

for all  $i$ .

To compute the cohomology groups  $H^i(\mathbb{P}(\mathcal{U}_x), f^* \mathcal{W}_\xi(L_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1))$  we use Hecke transformations.

A point in  $\mathbb{P}(\mathcal{U}_x)$  represents a vector bundle  $E$  and a line  $l$  in the fiber  $E_x$  at  $x$ , or equivalently a non-trivial exact sequence

$$(8) \quad 0 \longrightarrow E \longrightarrow F \longrightarrow \mathbb{C}_x \longrightarrow 0$$

determined up to a scalar multiple; here  $\mathbb{C}_x$  denotes the torsion sheaf supported at  $x$  with stalk  $\mathbb{C}$ . The sequences (8) fit together to form a universal sequence

$$(9) \quad 0 \longrightarrow (\mathrm{Id}_X \times f)^* \mathcal{U}_\xi \otimes p_{\mathbb{P}(\mathcal{U}_x)}^* \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) \longrightarrow \mathcal{F} \longrightarrow p_X^* \mathbb{C}_x \longrightarrow 0$$

on  $X \times \mathbb{P}(\mathcal{U}_x)$ . If  $\eta$  denotes the line bundle  $\xi \otimes \mathcal{O}_X(x)$  over  $X$  and  $\mathcal{M}_\eta$  the moduli space of stable bundles  $\mathcal{M}_\eta(n, d+1)$  then from (8) and (9) we get a rational map

$$\gamma : \mathbb{P}(\mathcal{U}_x) \dashrightarrow \mathcal{M}_\eta$$

which sends any pair  $(E, l)$  to  $F$ . This map is not everywhere defined since the bundle  $F$  in (8) need not be stable.

Our next object is to find a Zariski-open subset  $Z$  of  $\mathcal{M}_\eta$ , over which  $\gamma$  is defined and is a projective fibration, such that the complement of  $Z$  in  $\mathcal{M}_\eta$  has codimension at least 4. The construction and calculations are similar to those of [16, Proposition 6.8], but our results do not seem to follow directly from that proposition.

As in [16, §8] or [17, §5], we define a bundle  $F$  to be  $(0, 1)$ -stable if, for every proper subbundle  $G$  of  $F$ ,

$$\frac{\deg G}{\mathrm{rk} G} < \frac{\deg F - 1}{\mathrm{rk} F}.$$

Clearly every  $(0, 1)$ -stable bundle is stable. We denote by  $Z$  the subset of  $\mathcal{M}_\eta$  consisting of  $(0, 1)$ -stable bundles.

**Lemma 3.1.** (i)  $Z$  is a Zariski-open subset of  $\mathcal{M}_\eta$  whose complement has codimension at least 4.

(ii)  $\gamma$  is a projective fibration over  $Z$  and  $\gamma^{-1}(Z)$  is a Zariski-open subset of  $\mathbb{P}(\mathcal{U}_x)$  whose complement has codimension at least 4.

*Proof.* (i) The fact that  $Z$  is Zariski-open is standard (see [17, Proposition 5.3]).

The bundle  $F \in \mathcal{M}_\eta$  of rank  $n$  and degree  $d+1$  fails to be  $(0, 1)$ -stable if and only if it has a subbundle  $G$  of rank  $r$  and degree  $e$  such that  $ne \geq r((d+1) - 1)$ , i.e.,

$$(10) \quad rd \leq ne.$$

By considering the extensions

$$0 \longrightarrow G \longrightarrow F \longrightarrow H \longrightarrow 0,$$

we can estimate the codimension of  $\mathcal{M}_\eta - Z$  and show that it is at least

$$(11) \quad \delta = r(n-r)(g-1) + (ne - r(d+1))$$

(compare the proof of [17, Proposition 5.4]). Note that, since  $(n, d) = 1$ , (10) implies that  $rd \leq ne - 1$ . Given that  $g \geq 3$ , we see that  $\delta < 4$  only if  $g = 3$ ,  $n = 2, 3$  or  $g = 4$ ,  $n = 2$ . These are exactly the cases that were excluded in the introduction.



(ii)  $\gamma^{-1}(Z)$  consists of all pairs  $(E, l)$  for which the bundle  $F$  in (8) is  $(0, 1)$ -stable. As in (i), this is a Zariski-open subset. It follows at once from (10) that, if  $F$  is  $(0, 1)$ -stable, then  $E$  is stable. So, if  $F \in Z$ , it follows from (8) that  $\gamma^{-1}(F)$  can be identified with the projective space  $\mathbb{P}(F_x^*)$ . Using the universal projective bundle on  $X \times \mathcal{M}_\eta$ , we see that  $\gamma^{-1}(Z)$  is a projective fibration over  $Z$  (not necessarily locally trivial).

Suppose now that  $(E, l)$  belongs to the complement of  $\gamma^{-1}(Z)$  in  $\mathbb{P}(\mathcal{U}_x)$ . This means that the bundle  $F$  in (8) is not  $(0, 1)$ -stable and therefore possesses a subbundle  $G$  satisfying (10). If  $G \subset E$ , this contradicts the stability of  $E$ . So there exists an exact sequence

$$0 \longrightarrow G' \longrightarrow G \longrightarrow \mathbb{C}_x \longrightarrow 0$$

with  $G'$  a subbundle of  $E$  of rank  $r$  and degree  $e - 1$ . Moreover, since  $G$  is a subbundle of  $F$ ,  $G'_x$  must contain the line  $l$ . For fixed  $r, e$ , these conditions determine a subvariety of  $\mathbb{P}(\mathcal{U}_x)$  of dimension at most

$$(r^2(g-1)+1) + ((n-r)^2(g-1)+1) - g + (r-1) + ((g-1)r(n-r) + (rd - n(e-1)) - 1).$$

Since  $\dim \mathbb{P}(\mathcal{U}_x) = n^2(g-1) - g + n$ , a simple calculation shows that the codimension is at least the number  $\delta$  given by (11). As in (i), this gives the required result.  $\square$

By Lemma 3.1(ii) and a Hartogs-type theorem (see [9, Theorem 3.8 and Proposition 1.11]) we have an isomorphism

$$(12) \quad H^i(\mathbb{P}(\mathcal{U}_x), f^*\mathcal{W}_\xi(L_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)) \cong H^i(\gamma^{-1}(Z), f^*\mathcal{W}_\xi(L_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)|_{\gamma^{-1}(Z)})$$

for  $i \leq 2$ .

Now let  $F \in Z$ . As in the proof of Lemma 3.1, we identify  $\gamma^{-1}(F)$  with  $\mathbb{P}(F_x^*)$  and denote it by  $\mathbb{P}$ . On  $X \times \mathbb{P}$  there is a universal exact sequence

$$(13) \quad 0 \longrightarrow \mathcal{E} \longrightarrow p_X^*F \longrightarrow p_{\mathbb{P}}^*\mathcal{O}_{\mathbb{P}}(1) \otimes p_X^*\mathbb{C}_x \longrightarrow 0.$$

The restriction of (13) to any point of  $\mathbb{P}$  is isomorphic to the corresponding sequence (8).

**Proposition 3.2.** *Let  $\mathcal{F}$  be defined by the universal sequence (9). Then*

$$\mathcal{F}|_{X \times \mathbb{P}} \cong p_X^*F \otimes p_{\mathbb{P}}^*\mathcal{O}_{\mathbb{P}}(-1).$$

*Proof.* Restricting (9) to  $X \times \mathbb{P}$  gives

$$0 \longrightarrow (\text{Id}_X \times f)^*\mathcal{U}_\xi \otimes p_{\mathbb{P}(\mathcal{U}_x)}^*\mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1)|_{X \times \mathbb{P}} \longrightarrow \mathcal{F}|_{X \times \mathbb{P}} \longrightarrow p_X^*\mathbb{C}_x \longrightarrow 0.$$

This must coincide with the universal sequence (13) up to tensoring by some line bundle lifted from  $\mathbb{P}$ . The result follows.  $\square$

Next we tensor (9) by  $p_X^* L_1$ , restrict it to  $X \times \gamma^{-1}(Z)$  and take the direct image on  $\gamma^{-1}(Z)$ . This gives

$$(14) \quad 0 \longrightarrow f^* \mathcal{W}_\xi(L_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) \Big|_{\gamma^{-1}(Z)} \longrightarrow p_{\mathbb{P}(\mathcal{U}_x)*}(\mathcal{F} \otimes p_X^* L_1) \Big|_{\gamma^{-1}(Z)} \longrightarrow \mathcal{O}_{\gamma^{-1}(Z)} \longrightarrow 0.$$

**Proposition 3.3.**  $R_{\gamma*}^i(p_{\mathbb{P}(\mathcal{U}_x)*}(\mathcal{F} \otimes p_X^* L_1) \Big|_{\gamma^{-1}(Z)}) = 0$  for all  $i$ .

*Proof.* It is sufficient to show that  $p_{\mathbb{P}(\mathcal{U}_x)*}(\mathcal{F} \otimes p_X^* L_1) \Big|_{\mathbb{P}}$  has trivial cohomology. By Proposition 3.2

$$\begin{aligned} p_{\mathbb{P}(\mathcal{U}_x)*}(\mathcal{F} \otimes p_X^* L_1) \Big|_{\mathbb{P}} &\cong p_{\mathbb{P}*}(p_X^* F \otimes p_X^* L_1 \otimes p_{\mathbb{P}}^* \mathcal{O}_{\mathbb{P}}(-1)) \\ &\cong H^0(X, F \otimes L_1) \otimes \mathcal{O}_{\mathbb{P}}(-1) \end{aligned}$$

and the result follows.  $\square$

**Corollary 3.4.**  $\mathcal{R}_{\gamma*}^i(f^* \mathcal{W}_\xi(L_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) \Big|_{\gamma^{-1}(Z)}) = 0$  for  $i \neq 1$ . Moreover,

$$\mathcal{R}_{\gamma*}^1(f^* \mathcal{W}_\xi(L_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) \Big|_{\gamma^{-1}(Z)}) \cong \mathcal{O}_Z.$$

*Proof.* This follows at once from (14) and Proposition 3.3.  $\square$

Now we are in a position to compute the cohomology groups of  $H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)$  for  $i = 1, 2$ .

**Proposition 3.5.**  $H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) = 0$  for any  $x \in X$ .

*Proof.* The combination of (6), (7) and (12) yields

$$H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) \cong H^2(\gamma^{-1}(Z), f^* \mathcal{W}_\xi(L_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) \Big|_{\gamma^{-1}(Z)}).$$

Using Corollary 3.4 and Lemma 3.1(i), the Leray spectral sequence for the map  $\gamma$  gives

$$\begin{aligned} H^2(\gamma^{-1}(Z), f^* \mathcal{W}_\xi(L_1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}(1) \Big|_{\gamma^{-1}(Z)}) &\cong H^1(Z, \mathcal{O}_Z) \\ &\cong H^1(\mathcal{M}_\eta, \mathcal{O}_{\mathcal{M}_\eta}) \end{aligned}$$

It is known that  $H^1(\mathcal{M}_\eta, \mathcal{O}_{\mathcal{M}_\eta}) = 0$  [6]. Therefore,

$$H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) = 0.$$

$\square$

*Proof of Proposition 2.4.* Proposition 2.4 is an immediate consequence of Proposition 3.5.  $\square$

**Proposition 3.6.** For any point  $x \in X$ ,  $\dim H^1(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) = 1$ .

*Proof.* As in the proof of Proposition 3.5 we conclude that

$$H^1(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) \cong H^0(\mathcal{M}_\eta, \mathcal{O}_{\mathcal{M}_\eta}).$$

Now  $\mathcal{M}_\eta$  is just the non-singular part of the moduli space of semistable bundles of rank  $n$  and determinant  $\eta$ , and the latter space is complete and normal. So  $\dim H^0(\mathcal{M}_\eta, \mathcal{O}_{\mathcal{M}_\eta}) = 1$ .  $\square$

**Remark 3.7.** Since the fibres of  $\gamma$  are projective spaces, we have  $\gamma_* \mathcal{O}_{\gamma^{-1}(Z)} \cong \mathcal{O}_Z$  and all the higher direct images of  $\mathcal{O}_{\gamma^{-1}(Z)}$  are 0. Hence

$$H^i(Z, \mathcal{O}_Z) \cong H^i(\gamma^{-1}(Z), \mathcal{O}_{\gamma^{-1}(Z)})$$

for all  $i$ . Similarly

$$H^i(\mathbb{P}(\mathcal{U}_x), \mathcal{O}_{\mathbb{P}(\mathcal{U}_x)}) \cong H^i(\mathcal{M}_\xi, \mathcal{O}_{\mathcal{M}_\xi}) = 0$$

for  $i > 0$  since  $\mathcal{M}_\xi$  is a smooth projective rational variety. It follows from the proof of Lemma 3.1 that, if we define  $\delta$  as in (11),

$$\delta \geq i + 2 \geq 3 \Rightarrow H^i(Z, \mathcal{O}_Z) = 0.$$

The proof of Proposition 3.5 now gives

$$\delta \geq i + 2 \geq 4 \Rightarrow H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) = 0.$$

This in turn implies that  $R^i = 0$ . Proposition 2.1 and the Leray spectral sequence of  $p_3$  (cf. (1) and (2)) now give

$$\delta \geq i + 3 \geq 5 \Rightarrow H^i(\mathcal{M}_\xi, \mathcal{W}_\xi(L_1) \otimes \mathcal{W}_\xi(L_2)^*) = 0.$$

In particular, taking  $i = 2$  and  $L_1 = L_2 = L$ , and using (11), we obtain

$$H^2(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(L))) = 0$$

except possibly when  $g = 3, n = 2, 3, 4$ ;  $g = 4, n = 2$ ;  $g = 5, n = 2$ .

#### 4. PROOF OF PROPOSITION 2.5

Let  $\Delta$  be the diagonal divisor in  $X \times X$ . Pull back the exact sequence

$$0 \longrightarrow \mathcal{O}(-\Delta) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

to  $X \times \mathcal{M}_\xi \times X$  and tensor it with  $p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*$ . Now, the direct image sequence for the projection  $p_3$  gives the following exact sequence over  $X$

$$\begin{aligned} \longrightarrow \mathcal{R}^i p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*(-\Delta)) &\longrightarrow \mathcal{R}^i \longrightarrow \mathcal{R}^i p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}) \\ (15) \quad &\longrightarrow \mathcal{R}^{i+1} p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*(-\Delta)) \longrightarrow \dots \end{aligned}$$

The following propositions will be used in computing the direct images  $\mathcal{R}^i$ .

**Proposition 4.1.** *For any  $L_1 \in J$ , the direct images of*

$$p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}$$

*have the following description:*

1.  $\mathcal{R}^0 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}) \cong L_1$
2.  $\mathcal{R}^1 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}) \cong L_1 \otimes TX$
3.  $\mathcal{R}^2 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}) = 0$

where  $TX$  is the tangent bundle of  $X$ .

*Proof.* Identifying  $\Delta$  with  $X$  we have

$$\mathcal{R}^i p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*|_{\Delta \times \mathcal{M}_\xi}) \cong \mathcal{R}^i p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes \mathcal{U}_\xi^*).$$

The proposition follows from a result of Narasimhan and Ramanan [16, Theorem 2] that says

$$(16) \quad H^i(\mathcal{M}_\xi, \mathcal{U}_x \otimes \mathcal{U}_x^*) \cong \begin{cases} \mathbb{C} & \text{if } i = 0, 1 \\ 0 & \text{if } i = 2. \end{cases}$$

For  $i = 0$  the isomorphism is given by the obvious inclusion of  $\mathcal{O}_{\mathcal{M}_\xi}$  in  $\mathcal{U}_x \otimes \mathcal{U}_x^*$  and therefore globalises to give  $\mathcal{R}^0 p_{X*}(\mathcal{U}_\xi \otimes \mathcal{U}_\xi^*) \cong \mathcal{O}_X$ . Similarly for  $i = 1$  the isomorphism is given by the infinitesimal deformation map of  $\mathcal{U}_\xi$  regarded as a family of bundles over  $\mathcal{M}_\xi$  parametrised by  $X$ ; this globalises to  $\mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes \mathcal{U}_\xi^*) \cong TX$ .  $\square$

Propositions 4.1, 2.3 and the exact sequence (15) together give the following exact sequence of direct images

$$(17) \quad \begin{aligned} 0 \longrightarrow L_1 \longrightarrow \mathcal{R}^1 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*(-\Delta)) \longrightarrow \mathcal{R}^1 \xrightarrow{\alpha} L_1 \otimes TX \\ \longrightarrow \mathcal{R}^2 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*(-\Delta)) \longrightarrow \dots \end{aligned}$$

For any  $x \in X$  we have the cohomology exact sequence

$$(18) \quad \begin{aligned} \longrightarrow H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*(-x)) \longrightarrow H^i(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) \\ \longrightarrow H^i(\mathcal{M}_\xi, \mathcal{U}_x \otimes \mathcal{U}_x^*) \otimes (L_1)_x \longrightarrow H^{i+1}(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*(-x)), \end{aligned}$$

where  $(L_1)_x$  is the fiber of  $L_1$  at  $x$ .

By (4),  $p_{X*}(\mathcal{U}_\xi \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) = 0$ . So the Leray spectral sequence for  $p_X$  gives

$$H^1(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) \cong H^0(X, \mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*))$$

and

$$H^1(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*(-x)) \cong H^0(X, \mathcal{R}^1 p_{1*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)(-x)).$$

Since  $\mathcal{U}_x$  is simple [16, Theorem 2], (18) gives the exact sequence

$$\begin{aligned} 0 \longrightarrow (L_1)_x \longrightarrow H^0(X, \mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)(-x)) \\ \longrightarrow H^0(X, \mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)) \longrightarrow \dots \end{aligned}$$

This implies that  $\mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)$  has torsion at  $x$ . Now from (16) we conclude that  $\mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)$  is a torsion sheaf, and hence

$$H^1(X, \mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)(-x)) = H^1(X, \mathcal{R}^1 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)) = 0.$$

The Leray spectral sequence for  $p_X$  now yields

$$(19) \quad H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) \cong H^0(X, \mathcal{R}^2 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*))$$

and

$$(20) \quad H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*(-x)) \cong H^0(X, \mathcal{R}^2 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)(-x)).$$

Now from (16) it follows that  $\mathcal{R}^2 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*)$  is a torsion sheaf, and from Proposition 3.5 and (19) that its space of sections is 0. So  $\mathcal{R}^2 p_{X*}(\mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) = 0$  and by (20)

$$H^2(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*(-x)) = 0.$$

So, we have the following proposition:

**Proposition 4.2.**  $\mathcal{R}^2 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*(-\Delta)) = 0.$

Now Proposition 2.5 is easy to derive.

*Proof of Proposition 2.5.* By Proposition 3.6

$$H^1(X \times \mathcal{M}_\xi, \mathcal{U}_\xi \otimes p_X^* L_1 \otimes p_{\mathcal{M}}^* \mathcal{U}_x^*) \cong \mathbb{C}.$$

Therefore,  $\mathcal{R}^1 = \mathcal{R}^1 p_{3*}(p_{12}^* \mathcal{U}_\xi \otimes p_1^* L_1 \otimes p_{23}^* \mathcal{U}_\xi^*)$  is a line bundle. Moreover Proposition 4.2 implies that the map  $\alpha : \mathcal{R}^1 \rightarrow L_1 \otimes TX$  in the exact sequence (17) is surjective. Therefore  $\alpha$  must be an isomorphism, and the proof is complete.  $\square$

Earlier Theorems 2.6 and 2.9 were proved assuming Propositions 2.4 and 2.5. Therefore Theorems 2.6 and 2.9 are now established.

## 5. FAMILY OF DEFORMATIONS

Fix a point  $x \in X$ . Let  $\mathcal{L}$  be the Poincaré line bundle over  $X \times J$  which is trivial on  $x \times J$ . It was proved in [2] that the family  $\widetilde{\mathcal{U}}_\xi := p_{12}^* \mathcal{L} \otimes p_{13}^* \mathcal{U}_\xi$  over  $X \times J \times \mathcal{M}_\xi$  is a complete family of deformations of  $\mathcal{U}_\xi$  parametrised by  $J$ . In this section we will prove that the direct image

$$\widetilde{\mathcal{W}} := p_{23*}(\widetilde{\mathcal{U}}_\xi)$$

is a complete family of deformations of  $\mathcal{W}_\xi(L)$  for any  $L \in J$ .

First note that  $\widetilde{\mathcal{W}}$  is locally free. Indeed,  $\mathcal{R}^i p_{23*}(\widetilde{\mathcal{U}}_\xi) = 0$  for  $i \neq 0$ . Moreover, for each  $L \in J$ ,

$$\widetilde{\mathcal{W}}|_{\{L\} \times \mathcal{M}_\xi} \cong \mathcal{W}_\xi(L).$$

In [4] it was proved that  $\mathcal{W}_\xi$  is stable with respect to the unique polarisation of  $\mathcal{M}_\xi$ . From the proof of the Theorem in [4] we can see that for any line bundle  $L \in J$ ,  $\mathcal{W}_\xi(L)$  is also stable. Since  $J$  is irreducible, the Hilbert polynomial  $P$  of  $\mathcal{W}_\xi(L)$  is well defined and independent of the choice of  $L$ .

Denote by  $\mathcal{M}_X^0(\mathcal{W}_\xi)$  the connected component of the moduli space of stable sheaves, with the same Hilbert polynomial as  $\mathcal{W}_\xi(\mathcal{O})$ , on  $\mathcal{M}_\xi$  containing  $\mathcal{W}_\xi(\mathcal{O})$ . As in [3] the determinant line bundle  $M$  on  $\mathcal{M}_X^0(\mathcal{W}_\xi)$  defines a polarisation on  $\mathcal{M}_X^0(\mathcal{W}_\xi)$ .

Define the morphism

$$\phi : J \longrightarrow \mathcal{M}_X^0(\mathcal{W}_\xi)$$

by  $L \mapsto \mathcal{W}_\xi(L)$ .

**Theorem 5.1.** *The morphism*

$$\phi : J \longrightarrow \mathcal{M}_X^0(\mathcal{W}_\xi)$$

*is an isomorphism of polarised varieties.*

*Proof.* By Corollary 2.7  $\phi$  is injective, so its image has dimension  $g$ . On the other hand, by Theorem 2.9, the Zariski tangent space of  $\mathcal{M}_X^0(\mathcal{W}_\xi)$  also has dimension  $g$  at every point of the image of  $\phi$ . It follows that  $\mathcal{M}_X^0(\mathcal{W}_\xi)$  is smooth of dimension  $g$  at every point of the image of  $\phi$ . Hence, by Zariski's Main Theorem,  $\phi$  is an isomorphism onto an open subset of  $\mathcal{M}_X^0(\mathcal{W}_\xi)$ . Finally  $J$  is complete and  $\mathcal{M}_X^0(\mathcal{W}_\xi)$  is connected and separated (since  $\mathcal{W}_\xi(\mathcal{O})$  is stable), so  $\phi$  is an isomorphism.

Let  $\zeta$  be the polarisation on  $\mathcal{M}_X^0(\mathcal{W}_\xi)$  given by the determinant line bundle [3, Section 4]. Let  $\Theta$  denote the principal polarisation on  $J$  defined by a theta divisor. We wish to show that the isomorphism  $\phi$  takes  $\zeta$  to a nonzero constant scalar multiple (independent of the curve  $X$ ) of  $\Theta$ .

Take any family of pairs  $(X, \xi)$ , where  $X$  is a connected non-singular projective curve of genus  $g$  and  $\xi$  is a line bundle on  $X$  of degree  $d > n(2g - 2)$ , parametrized by a connected space  $T$ . Consider the corresponding family of moduli spaces  $\mathcal{M}_X^0(\mathcal{W}_\xi)$  (respectively, Jacobians  $J$ ) over  $T$ , where  $(X, \xi)$  runs over the family. Using the map  $\phi$  an isomorphism between these two families is obtained. The polarisation  $\zeta$  (respectively,  $\Theta$ ) defines a constant section of the second direct image over  $T$  of the constant sheaf  $\mathbb{Z}$  over the family. It is known that for the general curve  $X$  of genus  $g$ , the Neron-Severi group of  $J$  is  $\mathbb{Z}$ . Therefore, for such a curve,  $\phi$  takes  $\zeta$  to a nonzero constant scalar multiple of  $\Theta$ . Since  $T$  is connected, if  $T$  contains a curve with  $NS(J) = \mathbb{Z}$ , then  $\phi$  takes  $\zeta$  to the same nonzero constant scalar multiple of  $\Theta$  for every curve in the family. Since the moduli space of smooth curves of genus  $g$  is connected, the proof is complete.  $\square$

**Corollary 5.2.** *The family  $\widetilde{\mathcal{W}}$  parametrised by  $J$  is complete. Moreover, the infinitesimal deformation map of this family at any point of  $J$  is an isomorphism.*

*Proof.* This follows at once from the theorem.  $\square$

**Remark 5.3.** In the proof of Theorem 5.1, smoothness follows from the fact that the dimension of  $\mathcal{M}_X^0(\mathcal{W}_\xi)$  is equal to the dimension of its Zariski tangent space. So we do not need to know that  $H^2(\mathcal{M}_\xi, \text{End}(\mathcal{W}_\xi(L))) = 0$  (see Remark 3.7) or even that  $\mathcal{M}_X^0(\mathcal{W}_\xi)$  is reduced.

Finally we have our Torelli theorem.

**Corollary 5.4.** *Let  $X$  and  $X'$  be two non-singular algebraic curves of genus  $g \geq 3$  and let  $\xi$  (respectively  $\xi'$ ) be a line bundle of degree  $d > n(2g - 2)$  on  $X$  (respectively  $X'$ ). If  $\mathcal{M}_X^0(\mathcal{W}_\xi) \cong \mathcal{M}_{X'}^0(\mathcal{W}_{\xi'})$  as polarised varieties then  $X \cong X'$ .*

*Proof.* This follows at once from Theorem 5.1 and the classical Torelli theorem.  $\square$

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